Thermodynamics of a Black Hole with Moon

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For a rotating black hole perturbed by a particle on the "corotating" circular orbit (angular velocity equal to that of the event horizon), the black hole remains in equilibrium in the sense that the perturbed event horizon is a Killing horizon of the helical Killing field. The associated surface gravity is constant over the horizon and should correspond to the physical Hawking temperature. We calculate the perturbation in surface gravity/temperature, finding it negative: the moon has a cooling effect on the black hole. We also compute the surface area/entropy, and find no change from the background Kerr value.

Introduction.— Hawking's spectacular realization [1] that black holes radiate at a temperature given by their horizon surface gravity was a watershed moment in theoretical physics, endowing the classical laws of black hole mechanics [2–4] with a genuine thermodynamical status. With the result established for stationary (and hence axisymmetric [2]) black holes, a natural step forward would be to develop a fully general theory of radiating black holes. However, even the identification of a classical notion of surface gravity for a dynamical black hole is problematic, with different proposals yielding different results, even in spherical symmetry [5]. The main difficulty is the lack of a horizon Killing field, whose existence is crucial to the standard notion.

In this Letter we study a *highly dynamical* black hole spacetime for which the surface gravity can, remarkably, be unambiguously defined: a Kerr black hole perturbed by a corotating moon. This system is neither stationary nor axisymmetric, as it contains a fast-moving particle emitting significant gravitational radiation. However, the circular-orbit assumption leads to a *helical* symmetry of the spacetime, and—crucially—the corotation ensures that the orbits of this symmetry coincide with those of the null geodesic generators of the horizon. Thus the event horizon is a Killing horizon, and the surface gravity κ may be defined in the usual way. Standard calculations then show that κ is constant on the horizon (a "zeroth law"), and we argue that the Hawking temperature of the perturbed black hole must still be given by $\hbar\kappa/2\pi$.

An explicit calculation reveals that the perturbation in surface gravity caused by the orbiting moon is given by the simple, closed-form expression

$$\delta\kappa = -4\pi\, j\, \frac{\omega_H}{A}\,, \tag{1}$$

where j is the conserved angular momentum of the particle, while ω_H and A are the angular velocity and cross-sectional area of the unperturbed horizon. The perturbation is negative, showing that the tidally-induced deformation of the black hole horizon has a *cooling* effect. Recalling that the surface gravity of a Kerr black hole decreases with increasing spin, a general picture emerges whereby deformation (whether rotationally, tidally, or otherwise induced) yields a decrease in temperature.

Some of the inspiration for our work comes from Ref. [6], where a zeroth law for black holes is established in the context of certain helically symmetric exact solutions. Such solutions, however, require unphysical incoming radiation to balance the emitted radiation, yielding standing waves whose energy content precludes asymptotic flatness [7–9]. This physical limitation aside, the lack of smooth asymptotics removes the preferred normalization of the Killing field, and the numerical value of the surface gravity is essentially free.

In this work we avoid those difficulties by using the approximation of a small perturbing moon. To linear order in the size and mass of the moon, gravitational radiation-reaction is absent and incoming radiation is not needed to preserve the helical symmetry. (Physically, our approximation is valid over timescales where backreaction is a small effect.) The spacetime is asymptotically flat at null infinity, where the Killing field may be normalized relative to the time direction of a stationary observer. While our corotating setup is not generic, it can be realized in nature,[1] and our results provide an example of a realistic, highly dynamical black hole spacetime whose Hawking temperature is well-defined.

The entropy of any black hole is proportional to the surface area of horizon cross-sections. For our spacetime, the helical symmetry implies that the area must be constant (independent of slice), as expected from the second law for a system in equilibrium. An explicit calculation shows that the perturbation from the background Kerr value vanishes:

$$\delta A = 0\,. \tag{2}$$

We derive the perturbation in area from a first law of mechanics that holds for our perturbation [Eq. (12) below], and obtain the perturbation in surface gravity from a Smarr formula that holds for our perturbed spacetime [Eq. (14) below]. Interestingly, the moon is seen to affect the temperature, but not the entropy, of the companion hole. It would be illuminating to account for these results from a microscopic point of view.

Our conventions are those of Ref. [11]. In particular, the metric signature is $(-+++)$ and we use "geometrized units" where $G=c=1$. Latin indices $a,b,\cdots$ are abstract, while Greek indices $\mu,\nu,\cdots$ are used for coordinate components in a particular coordinate system.

[1] Stellar-mass compact objects orbiting supermassive black holes are a main target of the planned, space-based gravitational-wave detector LISA [10].

Problem Setting.— We consider a binary system consisting of a black hole orbited by a much smaller moon (see Fig. 1). To obtain an approximate description of this physical system we imagine attaching a one-parameter family of spacetimes $g_{ab}(\lambda)$ to this solution, where the size and mass of the moon are taken to zero with the parameter λ. The true spacetime is then approximated by a Taylor expansion,

$$g_{ab}(\lambda) = g_{ab} + \lambda\, \delta g_{ab} + \mathscr{O}(\lambda^2)\,, \tag{3}$$

where the moon is not present at lowest order. Here we use the variational notation to describe perturbations, $\delta g \equiv \partial_\lambda g|_{\lambda=0}$, and employ the standard abuse of notation that both background and perturbed metrics are denoted by g_{ab}, with context removing any ambiguity. Similarly, the background and perturbed versions of quantities covariantly constructed from the metric will be denoted in the variational manner.

In Ref. [12], it was shown that for a general body suitably scaled to zero size and mass, the perturbation δg_{ab} obeys the linearized Einstein equation with point-particle source,

$$\delta G_{ab} = 8\pi\, \delta T_{ab} = 8\pi m \int_\gamma \mathrm{d}\tau\, \delta_4(x,y)\, u_a u_b\,, \tag{4}$$

where the curve γ is a timelike geodesic of the background. (Here, τ is the proper time and $u^a = \mathrm{d}y^a/\mathrm{d}\tau$ the four-velocity.) We emphasize that the use of a point particle is *not* a statement about the composition of our body, but rather a consequence of considering an arbitrary body in the limit of small size. The constant parameter m has the interpretation of the ADM mass of the moon as measured in its near-zone [12].

We choose our background metric g_{ab} to be the Kerr geometry of mass M and angular momentum J. The black hole horizon has surface area $A = 8\pi M^2(1+\Delta)$, angular velocity $2M\omega_H = \chi/(1+\Delta)$, and surface gravity $2M\kappa = \Delta/(1+\Delta)$, where $\chi \equiv J/M^2$ and $\Delta \equiv \sqrt{1-\chi^2}$. We denote the timelike Killing field (normalized to -1 at infinity) by t^a and the axial Killing field (with integral curves of parameter length 2π) by ϕ^a. We take the geodesic γ to be the (unique) equatorial, circular orbit of azimuthal frequency $\Omega = \omega_H$. From the analysis of Ref. [13] one may check that this orbit exists and is timelike for all values of $0 < \chi < 1$. However, the orbit is stable only for $\chi < \chi_{\max} \simeq 0.36$. We denote the conserved orbital quantities associated with t^a and ϕ^a by

$$e = -m u^a t_a = m\,(1 - 2v^2 + \chi v^3)\, f(v,\chi)\,, \tag{5a}$$

$$j = m u^a \phi_a = mM\,(1 - 2\chi v^3 + \chi^2 v^4)\, f(v,\chi)/v\,, \tag{5b}$$

where $v^3 \equiv M\omega_H/(1-\chi M\omega_H)$ and $f \equiv (1 - 3v^2 + 2\chi v^3)^{-1/2}$, and will refer to e and j as the energy and angular momentum of the particle, respectively.

A strategy for constructing the physically-relevant solution of Eq. (4) is given in Ref. [14]. One first solves the Teukolsky equation for the perturbed Weyl scalar $\delta\psi_0$, making a choice of no incoming radiation. A "radiative" metric perturbation $\delta g^{\mathrm{rad}}_{ab}$ is then constructed from $\delta\psi_0$ using the procedure developed in [15–19]. In a suitable gauge, the Boyer-Lindquist coordinate components $\delta g^{\mathrm{rad}}_{\mu\nu}$ depend on Boyer-Lindquist ϕ and t only in the combination $\phi - \omega_H t$, showing that $\delta g^{\mathrm{rad}}_{ab}$ preserves the helical symmetry of the source δT_{ab}.

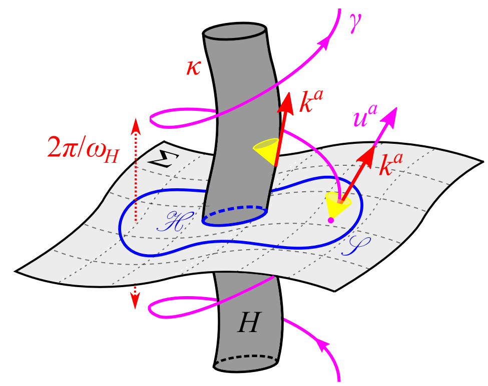

FIG. 1. Spacetime diagram depicting a black hole tidally perturbed by a corotating moon. (One spatial dimension is not shown.)

However, this perturbation does not satisfy (4) on its own, requiring an additional "nonradiative" piece $\delta g^{\mathrm{NR}}_{ab}$ to cancel a stationary, axisymmetric part of the source. Since $\delta g^{\mathrm{NR}}_{ab}$ must not change $\delta\psi_0$, it must agree with linearized Kerr away from the point particle, and may be matched at the source to ensure that the linearized Einstein equation is satisfied. (In particular, the non-radiative piece is stationary and axisymmetric, so that the entire metric perturbation is helically symmetric.) The only remaining freedom is that of a global linearized Kerr perturbation, which is fixed by the requirement that the spacetime have perturbed Bondi[2] mass and angular momentum given by $\delta M = e$ and $\delta J = j$, respectively. (Alternatively, one can demand that the nonradiative piece gives no contribution to the local Komar mass and angular momentum of the black hole, or equivalently that $\delta g^{\mathrm{NR}}_{ab}$ is pure gauge inside the particle orbit.) This choice ensures that the perturbation is "entirely due to the particle", with no spurious extra perturbation towards a nearby Kerr black hole. We will take this solution as our metric perturbation δg_{ab}, and we will use its general properties to establish our results.[3]

Zeroth Law.— Since our perturbed spacetime $g_{ab} + \lambda \delta g_{ab}$ becomes singular along the worldline γ, we cannot directly define the (future) event horizon H^+ as the boundary of the past of future null infinity. However, we may still employ this definition within the one-parameter family $g_{ab}(\lambda)$ at any finite λ, and it is clear on physical grounds that H^+ will be a smooth function of λ at $\lambda = 0$, ensuring that the perturbed event horizon is well-defined.[4] Likewise, the past horizon H^-

[2] Since the source has been radiating for all time, gravitational waves reach spatial infinity and the usual falloff conditions required for ADM quantities are not satisfied.

[3] There is a single property that we are unable to rigorously establish; we rely on a physical argument to assert that the angular velocity of the event horizon is unchanged by the corotating perturbation.

[4] If the small body is also a black hole, the event horizon will be disjoint for

should behave smoothly and meet H^+ on a regular bifurcation surface B. If we can establish that the expansion and shear of the horizons vanish to $\mathscr{O}(\lambda)$, then the local rigidity theorem of Ref. [20] implies, for our perturbed spacetime, the existence of a Killing field in a neighborhood of $H = H^+ \cup H^-$ that is tangent to the null generators of H, *i.e.*, that the event horizon is a Killing horizon. We may then appeal to [4], who establish the constancy of the surface gravity $\kappa^2 = \frac{1}{2}\nabla^a k^b \nabla_b k_a$ of any Killing horizon in any spacetime satisfying the dominant energy condition (at least locally). In particular, this establishes the constancy of $\delta\kappa$ for our locally vacuum spacetime.

It remains to show the vanishing of the perturbed expansion and shear. To do so, we introduce (at any $\lambda \geqslant 0$) a Newman-Penrose tetrad[5] $\{\ell^a, n^a, m^a, \bar{m}^a\}$ such that ℓ^a and n^a are tangent to the null generators of the future and past horizons, respectively, while m^a and $\bar{m}^a$ are parallel-transported along those generators. The expansion and shear are characterized on H^+ by the quantities $\rho = -m^a \bar{m}^b \nabla_b \ell_a$ and $\sigma = -m^a m^b \nabla_b \ell_a$, and on H^- by the corresponding quantities μ and λ. When evaluated on the future horizon, the Newman-Penrose (NP) equations for ρ and σ become

$$\ell^a \nabla_a \rho = \rho^2 + \sigma\bar{\sigma} + 2\varepsilon\rho \,, \tag{6a}$$

$$\ell^a \nabla_a \sigma = 2\rho\sigma + 2\varepsilon\sigma + \psi_0 \,, \tag{6b}$$

with $2\varepsilon = -n^a \ell^b \nabla_b \ell_a$ and $\psi_0 = C_{abcd}\ell^a m^b \ell^c m^d$. Equations (6) hold at finite λ on H^+. We now normalize the tetrad at $\lambda = 0$ such that $\ell^a = t^a + \omega_H \phi^a$ on the unperturbed future horizon. Then, 2ε coincides with the surface gravity κ of the Kerr spacetime. Futhermore, we have that ρ, σ, and ψ_0 all vanish when $\lambda = 0$, so that the perturbation of Eqs. (6) gives

$$(t^a + \omega_H \phi^a)\nabla_a \delta\rho = \kappa\,\delta\rho \,, \tag{7a}$$

$$(t^a + \omega_H \phi^a)\nabla_a \delta\sigma = \kappa\,\delta\sigma + \delta\psi_0 \,. \tag{7b}$$

(Here δ refers to a variation, rather than to the NP derivative operator, and κ refers to the surface gravity of the Kerr black hole, rather than to the NP spin coefficient.) However, the left-hand sides of Eqs. (7) vanish by the helical symmetry of the perturbed spacetime. Then, since we consider only $\chi < 1$ we have $\kappa \neq 0$, and it follows that

$$\delta\rho = 0 \quad \text{and} \quad \delta\sigma = -\kappa^{-1}\delta\psi_0 \,. \tag{8}$$

We now take advantage of the Teukolsky equation to compute $\delta\psi_0$ on the horizon. Equations (4.43), (4.40), and (4.42) of Ref. [21] show that each mode of $\delta\psi_0$, say $(\delta\psi_0)_{\ell m\omega}$, is given near the horizon ($r_* \to -\infty$) by

$$(\delta\psi_0)_{\ell m\omega} \sim A_{\ell m\omega}\, \mathrm{i}\tilde{k}\,(\tilde{k}^2 + \kappa^2)(-\mathrm{i}\tilde{k} + 2\kappa) \times {}_2S_{\ell m\omega}(\theta,\phi)\, e^{-\mathrm{i}(\tilde{k}r_* + \omega t)} \,, \tag{9}$$

where (t, r, θ, ϕ) are Boyer-Lindquist coordinates, with r_* the tortoise coordinate, ${}_2S_{\ell m\omega}(\theta,\phi)$ are spin-weighted spheroidal harmonics, $\tilde{k} \equiv \omega - m\omega_H$, and the amplitudes $A_{\ell m\omega}$ are determined by solving the Teukolsky equation. (Note that our $\delta\psi_0$ corresponds to their ψ_0^{HH}.) However, when a circular orbit of frequency ω_H is assumed, and no incoming radiation is chosen, the full field $\delta\psi_0$ is given by a sum over modes with $\omega = m\omega_H$, *i.e.*, we have $\tilde{k} = 0$. Then Eq. (9) gives $\delta\psi_0 = 0$ on H^+, and from (8) we conclude $\delta\sigma = 0$. A corresponding argument performed on the past horizon shows that $\delta\mu = \delta\lambda = 0$, and the demonstration of the vanishing of the perturbed expansion and shear is complete.

Horizon Killing Field.— The above argument establishes the existence of a horizon Killing field to $\mathscr{O}(\lambda)$, *i.e.*, of a vector field $k^a(\lambda)$ satisfying $\mathscr{L}_k g_{ab} = \mathscr{O}(\lambda^2)$ (at least in a neighborhood of the horizon) and normal to H. In addition to the helical Killing field of the metric perturbation (proportional to $t^a + \omega_H \phi^a$ in a gauge, such as that of Ref. [14], where the metric components $\delta g_{\mu\nu}$ are asymptotically vanishing), our perturbed spacetime also possesses the trivial Killing fields λt^a and $\lambda\phi^a$ inherited from the symmetries of the background. By a choice of normalization we may eliminate the perturbation to t^a, and the horizon Killing field can be written as

$$k^a(\lambda) = t^a + (\omega_H + \lambda\,\delta\omega_H)\,\phi^a + \mathscr{O}(\lambda^2) \,, \tag{10}$$

where $\delta\omega_H$ is a constant. Equation (10), together with the requirement of asymptotically vanishing metric components, defines $\delta\omega_H$ as an intrinsic, coordinate-invariant property of the perturbed spacetime. Since t^a and ϕ^a represent the time and rotational directions of a distant stationary observer, this constant can be interpreted as the perturbation in horizon angular velocity. Although we give no rigorous proof, we take it as obvious on physical grounds that $\delta\omega_H = 0$ for our perturbation. The interpretation of the choices made in defining δg_{ab} is that the perturbation is due entirely to the particle itself; and since the particle orbits at an angular frequency equal to that of the unperturbed horizon, it is clear that the horizon angular velocity ω_H cannot be changed.[6] Thus the horizon Killing field agrees with the helical Killing field, *i.e.*, $k^a = t^a + \omega_H \phi^a$.

Hawking Temperature.— We now argue that the horizon surface gravity κ of our tidally perturbed black hole still coincides with the physical Hawking temperature T_{H}. Our main point is that all of the essential properties underlying the semi-classical calculation for Kerr are preserved in our spacetime. In particular, we have a horizon Killing field $k^a = t^a + \omega_H \phi^a$, infinitesimally related to that of Kerr, which is normalized so that $k^a t_a = -1$ at infinity. The main new complication with respect to the Kerr case is that t^a and ϕ^a are not separate symmetries of our perturbed spacetime. However, t^a remains an *asymptotic* time translation symmetry, which may be used to

all $\lambda > 0$. As $\lambda \to 0$ and the small hole disappears, however, it seems clear that the large horizon will behave smoothly, and it is the perturbation of this component of the horizon that we study.

[5] The real null vectors ℓ^a and n^a satisfy $\ell^a n_a = -1$, while the complex null vector m^a satisfies $m^a \bar{m}_a = 1$. All other inner products vanish.

[6] Note that had we made a different choice of total mass and angular momentum of our spacetime, corresponding to adding in an extra piece of Kerr, the perturbation $\delta\omega_H$ would not vanish.

define positive and negative frequency modes with respect to a distant stationary detector, and the usual wavepacket scattering experiment may still be posed. We expect that, just as in the Kerr case, the mixing of positive and negative frequency modes would be controlled by the surface gravity κ associated with the Killing field k^a, leading to a particle flux through the distant detector with characteristic temperature $T_{\rm H} = \hbar\kappa/2\pi$. The lack of separate stationarity and axisymmetry will make this flux time and angle-dependent (though it must respect the helical symmetry), which a detailed calculation would presumably characterize in terms of a suitable "greybody factor" modifying the Planck spectrum. One may draw an analogy with a rotating ellipsoidal hot body of uniform surface temperature T_B; locally the body radiates thermally, while a distant detector measures a time and angle-dependent flux with characteristic temperature T_B. Similarly, we would regard $T_{\rm H}$ as the physical temperature of the perturbed black hole.

Perturbed Surface Area.— Our formula (2) for the perturbation in area is simply established from a first law that holds for our point-particle perturbation. Iyer and Wald [22] have given a general derivation of the first law for vacuum perturbations that are asymptotically flat at spatial infinity. We follow their general strategy, while making appropriate modifications for our non-vacuum perturbation that is asymptotically flat at null infinity. Calculations similar to those performed there yield the following identity (see also Ref. [23]):

$$\frac{1}{16\pi}\int_{\partial\Sigma}(\delta Q_{ab} - k^c\Theta_{abc}) = \frac{1}{8\pi}\int_\Sigma \varepsilon_{abcd}\,\delta G^{de}k_e\,, \qquad (11)$$

where $Q_{ab} = -\varepsilon_{abcd}\nabla^c k^d$ is the Noether two-form associated with k^a, and $\Theta_{abc} = \varepsilon_{abcd}\,g^{de}g^{fh}(\nabla_e\delta g_{fh} - \nabla_f\delta g_{eh})$ the symplectic potential three-form of general relativity, with ε_{abcd} the natural volume element associated with g_{ab}. Here Σ is an arbitrary spacelike slice transverse to k^a, with boundary $\partial\Sigma$.

We now choose for the inner and outer boundaries of Σ the (background) bifurcation sphere B and an arbitrary sphere $\mathscr{S}$ at future null infinity, respectively. According to the general analysis of [24], the integral over $\mathscr{S}$ yields the perturbation in the Bondi quantity associated with the asymptotic symmetry $t^a + \omega_H\phi^a$, that is, $\delta M - \omega_H\delta J$. (See Eq. (26) therein, where the last term vanishes for our stationary background.) Iyer and Wald [22] show that the surface integral over B gives $\kappa\,\delta A/8\pi$, where δA is the perturbation of the cross-sectional surface area.[7] Finally, using Eq. (4) together with the colinearity of the four-velocity and the helical Killing field at the particle, $k^a = z u^a$, the integral over Σ evaluates to $mz = e - \omega_H j$, which is the conserved orbital quantity associated with the helical symmetry, also refered to as the "redshift observable" [25, 26]. Thus we obtain a first law of black hole mechanics for our perturbative spacetime,

$$\delta M - \omega_H\delta J = \frac{\kappa}{8\pi}\,\delta A + mz\,. \qquad (12)$$

Using $\delta M = e$ and $\delta J = j$, Eq. (12) immediately shows that the perturbation in surface area vanishes: $\delta A = 0$.

Perturbed Surface Gravity.— To derive the perturbation $\delta\kappa$ in horizon surface gravity we first obtain a Smarr formula for our perturbed spacetime. We start from the standard identity (see, *e.g.*, Refs. [11, 27])

$$\frac{1}{8\pi}\int_{\partial\Sigma} Q_{ab} = \frac{1}{4\pi}\int_\Sigma \varepsilon_{abcd}\,R^{de}k_e\,, \qquad (13)$$

which is valid for any Killing field in any spacetime, and evaluate each term to $\mathscr{O}(\lambda)$ in our perturbed, helically symmetric spacetime. Using $k^a = t^a + \omega_H\phi^a$, the integral over $\mathscr{S}$ evaluates to the combination $(M + \lambda\,\delta M) - 2\omega_H(J + \lambda\,\delta J)$ of the Bondi mass and angular momentum. Now, rather than using the bifurcation sphere B, we choose for the inner boundary of Σ a cross-section $\mathscr{H}$ of the future event horizon H^+. (Here we work with the perturbed spacetime, and so the integral is over the perturbed horizon.) As shown in Ref. [6], the integral over $\mathscr{H}$ gives $[\kappa A + \lambda\,\delta(\kappa A)]/4\pi$. Finally, using the Einstein equation (4), the volume integral yields $\lambda\,mz$. When $\lambda = 0$ we recover Smarr's formula $M - 2\omega_H J = \kappa A/4\pi$ for a Kerr hole [28]. At first order in λ, we have the perturbed Smarr formula

$$\delta M - 2\omega_H\delta J = \frac{\delta(\kappa A)}{4\pi} + mz\,. \qquad (14)$$

Using $\delta M = e$, $\delta J = j$, $\delta A = 0$, and $mz = e - \omega_H j$ then yields our main result, Eq. (1) above.

From the expressions of A, ω_H, and j as functions of the Kerr parameter $\chi = J/M^2$, the perturbation $\delta\kappa$ is found to be a monotonically decreasing function of χ, with $\delta\kappa \to 0$ as $\chi \to 0$ and $\delta\kappa \to -(1/\sqrt{3})\,m/M^2$ as $\chi \to 1$. In particular, since the surface gravity of an extremal Kerr black hole vanishes, the perturbation becomes dominant in that limit, signaling the breakdown of the perturbation expansion. This is consistent with the observation made in [4] that nearly-extremal black holes are "loosely bound," in the sense that a small perturbation can raise a large tide. For the last stable corotating circular orbit, we have $\delta\kappa/\kappa \simeq -0.3\,m/M$.

Key to our derivation of the formulas (1) and (2) is the use of Stokes' theorem, which relates the metric perturbation on the horizon to the metric perturbation at infinity. Alternatively, it would be interesting to recover these results *via* a local analysis of the perturbation in a neighborhood of the horizon itself.

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[7] Since the perturbed expansion vanishes for our spacetime [see Eq. (8)], the perturbed surface area is independent of the horizon cross-section. Equivalently, we could evaluate the left-hand side integral of Eq. (11) on an arbitrary cross-section, finding $\delta(\kappa A)/8\pi$ for the first term, and $-A\,\delta\kappa/8\pi$ for the second.

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